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On the L^p norm of the torsion function

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Abstract
Bounds are obtained for the L^p norm of the torsion function v_Ω, i.e. the solution of $-\Delta v = 1,\ v \in H_0^1(\Omega)$, in terms of the Lebesgue measure of Ω and the principal eigenvalue $\lambda_1(\Omega)$ of the Dirichlet Laplacian acting in $L^2(\Omega)$. We show that these bounds are sharp for $1 \le p \le 2$.

Keywords Torsion function · Dirichlet conditions · Finite Lebesgue measure · L^p norm

Mathematics Subject Classification 35J25 · 35P99 · 58J35

1 Introduction

Let Ω be a non-empty open set in Euclidean space $\mathbb{R}^m$ with boundary $\partial\Omega$. It is well-known [2,3] that if the bottom of the Dirichlet Laplacian defined by

$$\lambda_1(\Omega) = \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |D\varphi|^2}{\displaystyle\int_\Omega \varphi^2} \tag{1}$$

is bounded away from 0, then

$$-\Delta v = 1,\ v \in H_0^1(\Omega) \tag{2}$$

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has a unique solution denoted by v_Ω. The function v_Ω is non-negative, pointwise increasing in Ω, and satisfies,

$$\lambda_1(\Omega)^{-1} \le \|v_\Omega\|_{L^\infty(\Omega)} \le (4 + 3m\log 2)\lambda_1(\Omega)^{-1}. \tag{3}$$

The m-dependent constant in the right-hand side of (3) has subsequently been improved [9,16]. We denote the optimal constant in the right-hand side of (3) by

$$\mathfrak{F}_\infty = \sup\{\lambda_1(\Omega)\|v_\Omega\|_{L^\infty(\Omega)} : \Omega \text{ open in } \mathbb{R}^m,\ |\Omega| < \infty\}, \tag{4}$$

suppressing the m-dependence. The torsional rigidity of Ω is defined by

$$T_1(\Omega) = \int_\Omega v_\Omega.$$

It plays a key role in different parts of analysis. For example the torsional rigidity of a cross section of a beam appears in the computation of the angular change when a beam of a given length and a given modulus of rigidity is exposed to a twisting moment [1,14]. It also arises in the definition of gamma convergence [7] and in the study of minimal submanifolds [12]. Moreover, $T_1(\Omega)/|\Omega|$ equals $\mathbb{E}_x(\tau_\Omega)$, the expected lifetime τ_Ω of Brownian motion in Ω, when averaged with respect to the uniform distribution over all starting points $x \in \Omega$.

The torsion function has been studied extensively and numerous works have been written on this subject. We just mention the paper [6], and the references therein. There the Kohler–Jobin rearrangement technique has been applied to the p-torsional rigidity, involving the p-Laplacian, and its first Dirichlet eigenvalue.

A classical inequality, e.g. [14], asserts that the function F_1 defined on the open sets in $\mathbb{R}^m$ with finite Lebesgue measure

$$F_1(\Omega) = \frac{T_1(\Omega)\lambda_1(\Omega)}{|\Omega|} \tag{5}$$

satisfies

$$F_1(\Omega) \le 1. \tag{6}$$

Since Ω has finite Lebesgue measure $|\Omega|$, (3) implies that $v \in L^p(\Omega)$ for $1 \le p \le \infty$. Moreover $\lambda_1(\Omega)$ is in that case the principal eigenvalue of the Dirichlet Laplacian. Motivated by (5) and (6) we make the following

Definition 1 (i) For Ω open in $\mathbb{R}^m$ with $0 < |\Omega| < \infty$ and $1 \le p < \infty$,

$$F_p(\Omega) = \frac{T_p(\Omega)\lambda_1(\Omega)}{|\Omega|^{1/p}}, \tag{7}$$

where

$$T_p(\Omega) = \|v_\Omega\|_{L^p(\Omega)} = \left(\int_\Omega v_\Omega^p\right)^{1/p}. \tag{8}$$

(ii) For Ω open in $\mathbb{R}^m$ with $\lambda_1(\Omega) > 0$,

$$F_\infty(\Omega) = \|v_\Omega\|_{L^\infty(\Omega)}\lambda_1(\Omega).$$

It follows from the Faber–Krahn inequality that if $|\Omega| < \infty$ then $\lambda_1(\Omega) > 0$. The converse does not hold for if Ω is the union of infinitely many disjoint balls of radii 1 then $\lambda_1(\Omega) > 0$ but Ω has infinite measure. Note that $2T_2^2(\Omega)/|\Omega|$ equals the second moment of the expected lifetime of Brownian motion in Ω, when averaged with respect to the uniform distribution over all starting points $x \in \Omega$.

Note that $\Omega \mapsto T_p(\Omega)$ is increasing while $\Omega \mapsto \lambda_1(\Omega)$ and $\Omega \mapsto |\Omega|^{-1/p}$ are decreasing. It is straightforward to verify that F_p, $1 \le p \le \infty$ is invariant under homotheties. That is, if $\alpha > 0$, $\alpha\Omega = \{x \in \mathbb{R}^m : x/\alpha \in \Omega\}$, then $F_p(\alpha\Omega) = F_p(\Omega)$.

Our main results are the following.

Theorem 1 *Let Ω be an open set in $\mathbb{R}^m$, $m = 1, 2, 3, \ldots$ with $|\Omega| < \infty$.*

(i) *If $1 \le p \le q \le \infty$ then,*

$$F_p(\Omega) \le F_q(\Omega) \le \mathfrak{F}_\infty. \tag{9}$$

(ii) *If $1 \le p \le 2$ then,*

$$F_p(\Omega) \le F_1(\Omega)^{1/p} \le 1. \tag{10}$$

Definition 2 For $1 \le p \le \infty$,

(i)

$$\mathfrak{F}_p = \sup\{F_p(\Omega) : \Omega \text{ open in } \mathbb{R}^m, |\Omega| < \infty\}, \tag{11}$$

(ii)

$$\mathfrak{G}_p = \inf\{F_p(\Omega) : \Omega \text{ open in } \mathbb{R}^m, |\Omega| < \infty\},$$

(iii)

$$\mathfrak{F}_p^{\text{convex}} = \sup\{F_p(\Omega) : \Omega \text{ open, convex in } \mathbb{R}^m, |\Omega| < \infty\},$$

(iv)

$$\mathfrak{G}_p^{\text{convex}} = \inf\{F_p(\Omega) : \Omega \text{ open, convex in } \mathbb{R}^m, |\Omega| < \infty\}.$$

It was shown in [5] that $\mathfrak{G}_\infty = 1$.

Theorem 2 *If $m = 1, 2, 3, \ldots$, and if $1 \le p < \infty$, then*

(i)

$$\mathfrak{G}_p = 0.$$

(ii) *The mapping $p \mapsto \mathfrak{G}_p^{convex}$ is non-decreasing, and*

$$\mathfrak{G}_p^{convex} \ge 2^{-3}\pi^2 m^{-(m+2p)/p}\left(\frac{\Gamma(\frac{m}{2}+1)\Gamma(p+1)}{\Gamma(\frac{m}{2}+p+1)}\right)^{1/p}. \tag{12}$$

It follows from (12) that $\lim_{p\to\infty} \mathfrak{G}_p^{\text{convex}} \geq \pi^2/8$. This jibes with the result of [13] that

$$\mathfrak{G}_\infty^{\text{convex}} = \frac{\pi^2}{8}.$$

A monotone increasing sequence of cuboids which exhausts the open connected set bounded by two parallel $(m-1)$-dimensional hyperplanes is a minimising sequence for $\mathfrak{G}_\infty^{\text{convex}}$. See also Theorem 2 in [5].

Theorem 3 *Let* $m = 2, 3, \ldots$.

(i) *The mappings* $p \mapsto \mathfrak{F}_p$, *and* $p \mapsto \mathfrak{F}_p^{convex}$ *are non-decreasing on* $[1, \infty]$.

(ii) *If*

$$p_m = \inf\{p \geq 1 : \mathfrak{F}_p > 1\},$$

and

$$p_m^{convex} = \inf\{p \geq 1 : \mathfrak{F}_p^{convex} > 1\},$$

then

$$2 \leq p_m \leq p_m^{convex} \leq 8m. \tag{13}$$

In particular

$$\mathfrak{F}_p = 1,\ \ 1 \leq p \leq p_m.$$

(iii) *Formula* (11) *defining* $\mathfrak{F}_p$ *does not have a maximiser for* $1 \leq p \leq 2$. *The maximising sequence constructed in [4] for* $\mathfrak{F}_1$ *is also a maximising sequence for* $\mathfrak{F}_p$, $1 \leq p \leq p_m$. *Hence inequality* (10) *actually reads* $F_2(\Omega) \leq F_1(\Omega)^{1/p} < 1,\ 1 \leq p \leq 2$.

(iv) *The mappings* $p \mapsto \mathfrak{F}_p$, *and* $p \mapsto \mathfrak{F}_p^{convex}$ *are left-continuous on* $(1, \infty]$.

(v) *If* $n \in \mathbb{N},\ 1 \leq p$, *then*

$$\mathfrak{F}_{p+n} \leq \left(\frac{p+n}{4^n p}\prod_{j=1}^{n}(p+j)\right)^{\frac{1}{p+n}} \mathfrak{F}_p^{\frac{p}{p+n}}, \tag{14}$$

$$\mathfrak{F}_{p+n}^{convex} \leq \left(\frac{p+n}{4^n p}\prod_{j=1}^{n}(p+j)\right)^{\frac{1}{p+n}} \left(\mathfrak{F}_p^{convex}\right)^{\frac{p}{p+n}}. \tag{15}$$

In particular if $1 \leq p \leq 2$, *then*

$$\mathfrak{F}_{p+1} \leq \left(\frac{(p+1)^2}{4p}\right)^{\frac{1}{p+1}}. \tag{16}$$

(vi)

$$\mathfrak{F}_n \leq \left(\frac{n.n!}{4^{n-1}}\right)^{\frac{1}{n}},\ n \in \mathbb{N}, \tag{17}$$

and $\mathfrak{F}_3 \leq 3^{2/3}/2 = 1.04004\ldots$.

(vii) $p \mapsto \mathfrak{F}_p$ *is differentiable at* $p = 2$, *with* $\mathfrak{F}_2' = 0$.
(viii) *If* $1 \le p \le 2$, *then*

$$\mathfrak{F}_p^{convex} \le \left(\mathfrak{F}_1^{convex}\right)^{1/p}. \tag{18}$$

(ix) *For* $m = 2$,

$$p_2^{\text{convex}} \ge 2.0186.$$

This paper is organised as follows. In Sect. 2 we prove Theorems 1 and 2. The proof of Theorem 3 will be given in Sect. 3.

We note that a general multiplicative inequality involving $T_p(\Omega)$, $\lambda_1(\Omega)$ and $|\Omega|$ will involve three exponents. However, the requirement that it be invariant under homotheties reduces the number of exponents to two. In Sect. 4 we briefly discuss this two-parameter family of inequalities, and determine which parameter pair yields a finite supremum.

2 Proofs of Theorems 1, 2

Proof of Theorem 1 (i) To prove (9) for $1 \le p \le q < \infty$ we use Hölder's inequality to obtain that

$$\int_\Omega v_\Omega^p \le \left(\int_\Omega v_\Omega^q\right)^{p/q} |\Omega|^{(q-p)/q}.$$

So we have that

$$\|v_\Omega\|_{L^p(\Omega)} \le \|v_\Omega\|_{L^q(\Omega)} |\Omega|^{\frac{1}{p}-\frac{1}{q}}.$$

This, together with (7), implies (9). In case $q = \infty$,

$$\|v_\Omega\|_{L^p(\Omega)} \le \|v_\Omega\|_{L^\infty(\Omega)} |\Omega|^{1/p}.$$

(ii) To prove (10) we observe that since Ω has finite Lebesgue measure the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ is discrete, and consists of an increasing sequence of eigenvalues

$$\{\lambda_1(\Omega) \le \lambda_2(\Omega) \le \lambda_3(\Omega) \le \dots .\},$$

accumulating at infinity, where we have included multiplicities. We denote a corresponding orthonormal basis of eigenfunctions by $\{\varphi_{j,\Omega}, j = 1, 2, 3, \dots\}$. The resolvent of the Dirichlet Laplacian acting in $L^2(\Omega)$ is compact, and its kernel H_Ω has an L^2-eigenfunction expansion given by

$$H_\Omega(x, y) = \sum_{j=1}^\infty \frac{1}{\lambda_j(\Omega)} \varphi_{j,\Omega}(x)\varphi_{j,\Omega}(y).$$

So v_Ω, defined by (2), is given by

$$v_\Omega(x) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j(\Omega)} \left(\int_\Omega \varphi_{j,\Omega} \right) \varphi_{j,\Omega}(x).$$

Since $v_\Omega \in L^2(\Omega)$ we have by orthonormality that

$$\begin{aligned}
\int_\Omega v_\Omega^2 &= \int_\Omega dx \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{\lambda_j(\Omega)} \left(\int_\Omega \varphi_{j,\Omega} \right) \varphi_{j,\Omega}(x) \frac{1}{\lambda_k(\Omega)} \left(\int_\Omega \varphi_{k,\Omega} \right) \varphi_{k,\Omega}(x) \\
&= \sum_{j=1}^{\infty} \frac{1}{\lambda_j^2(\Omega)} \left(\int_\Omega \varphi_{j,\Omega} \right)^2 \\
&\le \frac{1}{\lambda_1(\Omega)} \sum_{j=1}^{\infty} \frac{1}{\lambda_j(\Omega)} \left(\int_\Omega \varphi_{j,\Omega} \right)^2 \\
&= \frac{T_1(\Omega)}{\lambda_1(\Omega)}.
\end{aligned} \tag{19}$$

We conclude that

$$T_2(\Omega) \le \left(\frac{T_1(\Omega)}{\lambda_1(\Omega)} \right)^{1/2}. \tag{20}$$

Multiplying both sides of the inequality above with $\lambda_1(\Omega)/|\Omega|^{1/2}$ we obtain that $F_2(\Omega) \le \big(F_1(\Omega)\big)^{1/2}$. By (i) $\big(F_1(\Omega)\big)^{1/2} \le \big(F_2(\Omega)\big)^{1/2}$. This, together with the previous inequality, implies that $F_2(\Omega) \le 1$. We now use Hölder's inequality, and interpolate with $0 < \alpha < 1$, $\rho > 1$ as follows.

$$\int_\Omega v_\Omega^p = \left(\int_\Omega v_\Omega^{\alpha p \rho} \right)^{1/\rho} \left(\int_\Omega v_\Omega^{(1-\alpha) p\rho/(\rho-1)} \right)^{(\rho-1)/\rho}.$$

Choosing $\alpha p \rho = 2$, $(1-\alpha)p\rho/(\rho-1) = 1$ gives that $\rho = (p-1)^{-1}$. Hence by (20),

$$T_p^p(\Omega) = \int_\Omega v_\Omega^p \le \left(T_2^2(\Omega) \right)^{p-1} \left(T_1(\Omega) \right)^{2-p} \le \frac{T_1(\Omega)}{\lambda_1(\Omega)^{p-1}}.$$

Multiplying both sides of the inequality above with $\lambda_1(\Omega)^p/|\Omega|$ gives that

$$F_p^p(\Omega) \le F_1(\Omega).$$

□

Proof of Theorem 2 (i) We let Ω_n be the disjoint union of one ball of radius 1 and n balls with radii r_n, with $r_n < 1$. Then

$$|\Omega_n| = \big(n r_n^m + 1\big)|B_1|,$$

where $B_1 = \{x \in \mathbb{R}^m : |x| < 1\}$. Since $r_n < 1$ we have that

$$\lambda_1(\Omega_n) = \lambda_1(B_1).$$

Since T_p^p is additive on disjoint open sets we have by scaling that

$$T_p^p(\Omega_n) = \big(nr_n^{2p+m} + 1\big)T_p^p(B_1).$$

Therefore

$$\begin{aligned} F_p^p(\Omega_n) &= \frac{\big(nr_n^{2p+m} + 1\big)T_p^p(B_1)\lambda_1^p(B_1)}{\big(nr_n^m + 1\big)|B_1|} \\ &= \frac{nr_n^{2p+m} + 1}{nr_n^m + 1} F_p^p(B_1) \\ &\le \big(r_n^{2p} + n^{-1}r_n^{-m}\big)F_p^p(B_1). \end{aligned} \tag{21}$$

We now choose r_n as to minimise the right-hand side of (21),

$$r_n = \left(\frac{m}{2pn}\right)^{1/(2p+m)}.$$

This gives that

$$F_p^p(\Omega_n) \le \left(1 + \frac{2p}{m}\right)\left(\frac{m}{2p}\right)^{2p/(2p+m)} n^{-2p/(2p+m))} F_p^p(B_1),$$

which implies the assertion.

(ii) The first part of the assertion follows directly by (9). To prove the second part we recall John's ellipsoid theorem [10,11] which asserts the existence of an ellipsoid Υ with centre c such that $\Upsilon \subset \Omega \subset c + m(\Upsilon - c)$. Here $c + m(\Upsilon - c) = \{c + m(x - c) : x \in \Upsilon\}$. This is the dilation of Υ by the factor m. Υ is the ellipsoid of maximal volume in Ω. By translating both Ω and Υ we may assume that

$$\Upsilon = \left\{x \in \mathbb{R}^m : \sum_{i=1}^m \frac{x_i^2}{a_i^2} < 1\right\}, \qquad a_i > 0, \quad i = 1, \dots, m.$$

It is easily verified that the unique solution of (2) for Υ is given by

$$v_\Upsilon(x) = 2^{-1}\left(\sum_{i=1}^m \frac{1}{a_i^2}\right)^{-1}\left(1 - \sum_{i=1}^m \frac{x_i^2}{a_i^2}\right).$$

By changing to spherical coordinates, we find that

$$\int_{\Upsilon} v_{\Upsilon}^{p} = 2^{-p}\omega_m \frac{\Gamma(\frac{m}{2}+1)\Gamma(p+1)}{\Gamma(\frac{m}{2}+p+1)} \left(\sum_{i=1}^{m} \frac{1}{a_i^2}\right)^{-p} \prod_{i=1}^{m} a_i,$$

where $\omega_m = |B_1|$. Since $\Omega \mapsto v_\Omega$ is increasing we have by (8) that $\Omega \mapsto T_p(\Omega)$ is increasing, and

$$\begin{aligned} T_p(\Omega) &\ge T_p(\Upsilon) \\ &= 2^{-1}\omega_m^{1/p} \left(\frac{\Gamma(\frac{m}{2}+1)\Gamma(p+1)}{\Gamma(\frac{m}{2}+p+1)}\right)^{1/p} \left(\sum_{i=1}^{m} \frac{1}{a_i^2}\right)^{-1} \left(\prod_{i=1}^{m} a_i\right)^{1/p}. \end{aligned} \tag{22}$$

Since $\Omega \subset m\Upsilon$,

$$|\Omega| \le \int_{m\Upsilon} dx = \omega_m m^m \prod_{i=1}^{m} a_i. \tag{23}$$

By the monotonicity of Dirichlet eigenvalues, we have that $\lambda_1(\Omega) \ge \lambda_1(m\Upsilon)$. The ellipsoid $m\Upsilon$ is contained in a cuboid with lengths $2ma_1, \dots, 2ma_m$. So we have that

$$\lambda_1(\Omega) \ge \frac{\pi^2}{4m^2} \sum_{i=1}^{m} \frac{1}{a_i^2}. \tag{24}$$

Combining (22), (23), (24), and (8) gives (12). □

3 Proof of Theorem 3

(i) It follows from the second inequality in (9) that $\mathfrak{F}_q \le \mathfrak{F}_\infty$. Hence $F_p(\Omega) \le \mathfrak{F}_q \le \mathfrak{F}_\infty$. Taking subsequently the supremum over all Ω with finite measure we obtain the first assertion under (i). As (9) holds for all open sets with finite measure, it also holds for all bounded convex sets. Then, the preceding argument gives the second assertion under (i).
(ii) It follows from (10) that $\mathfrak{F}_p \le 1,\ 1 \le p \le 2$. In Theorem 1.2 of [4] it was shown that the bound $F_1(\Omega) \le 1$ is sharp. That is $\mathfrak{F}_1 = 1$. This, together with (i), then implies that $\mathfrak{F}_p = 1$ for $1 \le p \le 2$. Hence $p_m \ge 2$. Since $\mathfrak{F}_p^{\text{convex}} \le \mathfrak{F}_p$ we conclude the second inequality in (13). To prove the upper bound on p_m^{convex} we recall that

$$v_{B_1}(x) = \frac{1-|x|^2}{2m}.$$

Hence, denoting by $(f)_+$ the positive part of a real-valued function f, we have that

$$
\begin{aligned}
T_p(B_1) &= \left(\int_{[0,1]} dr\, m\omega_m \left(\frac{1-r^2}{2m}\right)^p r^{m-1}\right)^{1/p} \\
&= \frac{(m\omega_m)^{1/p}}{2^{(p+1)/p} m} \left(\int_{[0,1]} d\theta (1-\theta)^p \theta^{(m-2)/2}\right)^{1/p} \\
&\geq \frac{(m\omega_m)^{1/p}}{2^{(p+1)/p} m} \left(\int_{[0,1]} d\theta (1-p\theta)_+ \theta^{(m-2)/2}\right)^{1/p} \\
&\geq \frac{2^{1/p}\omega_m^{1/p}}{2m(m+2)^{1/p} p^{m/(2p)}} \\
&\geq \frac{\omega_m^{1/p}}{2m^{(p+1)/p} p^{m/(2p)}}.
\end{aligned} \tag{25}
$$

It follows that

$$
\mathfrak{F}_p \geq F_p(B_1) \geq \frac{j^2_{(m-2)/2}}{2m^{(p+1)/p} p^{m/(2p)}},
$$

where $\lambda_1(B_1) = j^2_{(m-2)/2}$, and $j_{(m-2)/2}$ is the first positive zero of the Bessel function $J_{(m-2)/2}$. Hence

$$
\mathfrak{F}_{8m} \geq \frac{j^2_{(m-2)/2}}{2m^{1+\frac{1}{8m}}(8m)^{\frac{1}{16}}} \geq \frac{j^2_{(m-2)/2}}{2^{\frac{19}{16}} m^{\frac{9}{8}}}. \tag{26}
$$

One verifies numerically that for $m = 2, \dots, 19$, the right-hand side of (26) is strictly greater than 1. Since $j^2_{(m-2)/2} \geq ((m-2)/2)^2$ (see inequality (1.6) in [8]) we have for $m \geq 20$ that $j^2_{(m-2)/2} > m^2/5$. But $m^{\frac{7}{8}} \geq 5 \cdot 2^{19/16}$, $m \geq 20$.
(iii) It was shown in [4] that the formula defining $\mathfrak{F}_1$ in (11) does not have a maximiser. Since by (10), $F_p(\Omega) \leq F_1(\Omega)^{1/p} \leq 1$ for any $1 \leq p \leq 2$ and any open subset $\Omega \subset \mathbb{R}^m$ with $|\Omega| < \infty$, none of the formulae defining $\mathfrak{F}_p$, $1 \leq p \leq 2$, have maximisers. Clearly, the maximising sequence constructed in [4] for $\mathfrak{F}_1$ is a maximising sequence for $\mathfrak{F}_p$, $1 \leq p \leq p_m$.
(iv) To prove left-continuity we first fix $1 < q < \infty$, and let $\epsilon > 0$ be arbitrary. There exists an open set $\Omega_{q,\epsilon} \subset \mathbb{R}^m$ such that

$$
\mathfrak{F}_q \geq F_q(\Omega_{q,\epsilon}) \geq \mathfrak{F}_q - \frac{\epsilon}{2}. \tag{27}
$$

By scaling we may assume that $|\Omega_{q,\epsilon}| = 1$. Let $p \in [1, q)$. Then

$$
\mathfrak{F}_q \geq \mathfrak{F}_p \geq F_p(\Omega_{q,\epsilon}),
$$

and

$$\int_{\Omega_{q,\epsilon}} v_{\Omega_{q,\epsilon}}^q \le \|v_{\Omega_{q,\epsilon}}\|_{L^\infty(\Omega_{q,\epsilon})}^{q-p} \int_{\Omega_{q,\epsilon}} v_{\Omega_{q,\epsilon}}^p \\ \le \mathfrak{F}_\infty^{q-p} \lambda_1^{p-q}(\Omega_{q,\epsilon}) \int_{\Omega_{q,\epsilon}} v_{\Omega_{q,\epsilon}}^p,$$

implying that

$$F_p(\Omega_{q,\epsilon}) \ge \mathfrak{F}_\infty^{(p-q)/p} F_q(\Omega_{q,\epsilon})^{q/p}.$$

Since $p \mapsto F_p$ is increasing we have that

$$\lim_{p \uparrow q} F_p(\Omega_{q,\epsilon}) \ge \mathfrak{F}_\infty^{(p-q)/p} F_q(\Omega_{q,\epsilon})^{q/p}. \tag{28}$$

Since $p < q$, we have by the continuity of the right-hand side of (28) in p, (26) and (27) that

$$\mathfrak{F}_q \ge \lim_{p \uparrow q} \mathfrak{F}_p \ge F_q(\Omega_{q,\epsilon}) \ge \mathfrak{F}_q - \frac{\epsilon}{2}.$$

Letting $\epsilon \downarrow 0$ concludes the proof for $1 < q < \infty$.

To prove left-continuity at $q = \infty$ we let $\epsilon > 0$ be arbitrary. By (11) there exists an open set $\Omega_{\infty,\epsilon}$ such that

$$\mathfrak{F}_\infty \ge F_\infty(\Omega_{\infty,\epsilon}) \ge \mathfrak{F}_\infty - \frac{\epsilon}{2}. \tag{29}$$

Without loss of generality we may assume by scaling that $|\Omega_{\infty,\epsilon}| = 1$. Then $v_{\Omega_{\infty,\epsilon}} \in L^p(\Omega_{\infty,\epsilon}),\ 1 \le p \le \infty$, and

$$F_\infty(\Omega_{\infty,\epsilon}) = \lim_{p \to \infty} F_p(\Omega_{\infty,\epsilon}).$$

Hence there exists $p(\epsilon) < \infty$ such that

$$|F_\infty(\Omega_{\infty,\epsilon}) - F_p(\Omega_{\infty,\epsilon})| \le \frac{\epsilon}{2}, \quad p \ge p(\epsilon). \tag{30}$$

This implies, by (29) and (30), that

$$\begin{aligned} \mathfrak{F}_p &\ge F_p(\Omega_{\infty,\epsilon}) \\ &\ge F_\infty(\Omega_{\infty,\epsilon}) - \frac{\epsilon}{2} \\ &\ge \mathfrak{F}_\infty - \epsilon, \quad p \ge p(\epsilon). \end{aligned} \tag{31}$$

Hence by (i) and (31),

$$\mathfrak{F}_\infty \ge \lim_{p \uparrow \infty} \mathfrak{F}_p \ge \mathfrak{F}_\infty - \epsilon.$$

The left-continuity at ∞ now follows since $\epsilon > 0$ was arbitrary.

(v) Let $n \in \mathbb{N},\ p \geq 1$. Without loss of generality we may assume that $|\Omega| = 1$. An integration by parts shows that

$$\int_\Omega v_\Omega^p = -\int_\Omega v_\Omega^p \Delta v_\Omega = p\int_\Omega v_\Omega^{p-1}|Dv_\Omega|^2 = \frac{4p}{(p+1)^2}\int_\Omega |Dv_\Omega^{(p+1)/2}|^2. \quad (32)$$

By (1)

$$\lambda_1(\Omega) \leq \frac{\int_\Omega |Dv_\Omega^{(p+1)/2}|^2}{\int_\Omega v_\Omega^{p+1}}. \quad (33)$$

By (32) and (33) we have that

$$\int_\Omega v_\Omega^p \geq \frac{4p}{(p+1)^2}\lambda_1(\Omega)\int_\Omega v_\Omega^{p+1}. \quad (34)$$

Multiplying both sides of (34) by $\lambda_1^p(\Omega)$ gives that

$$\frac{4p}{(p+1)^2}F_{p+1}^{p+1}(\Omega) \leq F_p^p(\Omega). \quad (35)$$

Taking the supremum over all open $\Omega \subset \mathbb{R}^m$ with measure 1, in the right-hand side of (35), and subsequently in the left-hand side of (35) gives that

$$\mathfrak{F}_{p+1}^{p+1} \leq \frac{(p+1)^2}{4p}\mathfrak{F}_p^p. \quad (36)$$

Iterating (36) $n-1$ times we find (14). The same calculation carries over when Ω is an open, bounded convex set. This proves (15). By part (ii) we have that for $1 \leq p \leq p_m,\ \mathfrak{F}_p = 1$. This, together with (14), gives (16).
(vi) Since $\mathfrak{F}_1 = 1$, we consider the case $n \in N,\ n \geq 2$. Put $p = 1$ in (14), and replace n by $n-1$. This gives (17).
(vii) Substituting $p = 1 + \delta,\ 0 < \delta \leq 1$ in (16) gives that

$$\delta^{-1}\big(\mathfrak{F}_{2+\delta} - \mathfrak{F}_2\big) \leq \delta^{-1}\left(\left(1 + \frac{\delta^2}{4}\right)^{\frac{1}{2+\delta}} - 1\right),$$

and the assertion follows by L' Hôpital's rule.
(viii) Taking suprema in (10) over all bounded convex open sets Ω yields (18).
(ix) Let $m = 2$. By (18), and the numerical estimate (1.10) in [4] we have for $p = 1 + \delta, 0 < \delta \leq 1$, that

$$\mathfrak{F}_{1+\delta}^{\text{convex}} \leq \left(1 - \frac{1}{11560}\right)^{\frac{1}{1+\delta}}. \quad (37)$$

By (15) for $n = 1$ we have that

$$\begin{aligned}
\mathfrak{F}_{2+\delta}^{\text{convex}} &\le \left(\frac{(2+\delta)^2}{4+4\delta}\right)^{\frac{1}{2+\delta}} \left(\mathfrak{F}_{1+\delta}^{\text{convex}}\right)^{\frac{1+\delta}{2+\delta}} \\
&\le \left(1+\frac{\delta^2}{4}\right)^{\frac{1}{2+\delta}} \left(\mathfrak{F}_{1+\delta}^{\text{convex}}\right)^{\frac{1+\delta}{2+\delta}} \\
&\le \left(1+\frac{\delta^2}{4}\right)^{\frac{1}{2+\delta}} \left(1-\frac{1}{11560}\right)^{\frac{1}{2+\delta}},
\end{aligned} \tag{38}$$

where we have used (37) in the last inequality. Since the right-hand side of (38) is equal to 1 for $\delta^* = \frac{2}{(11559)^{1/2}}$, we conclude that

$$p_2^{\text{convex}} \ge 2 + \delta^*,$$

which proves the assertion in (ix). □

4 A two-parameter family of inequalities

As mentioned at the end of the Introduction one can define a two-parameter family of products involving $T_p(\Omega)$, $\lambda_1(\Omega)$, and $|\Omega|$, which is invariant under homotheties.

Definition 3 For an open set $\Omega \subset \mathbb{R}^m$ with finite Lebesgue measure, $p \ge 1$, $q \in \mathbb{R}$,

(i)
$$F_{p,q}(\Omega) = \frac{T_p(\Omega)\lambda_1^q(\Omega)}{|\Omega|^{\frac{1}{p}+\frac{2}{m}(1-q)}}, \tag{39}$$

(ii)
$$F_{\infty,q}(\Omega) = \frac{\|v_\Omega\|_{L^\infty(\Omega)}\lambda_1^q(\Omega)}{|\Omega|^{\frac{2}{m}(1-q)}}, \tag{40}$$

(iii)
$$\mathfrak{F}_{p,q} = \sup\{F_{p,q}(\Omega) : \Omega \text{ open in } \mathbb{R}^m,\ |\Omega| < \infty\}, \tag{41}$$

(iv)
$$\mathfrak{F}_{\infty,q} = \sup\{F_{\infty,q}(\Omega) : \Omega \text{ open in } \mathbb{R}^m,\ |\Omega| < \infty\}. \tag{42}$$

It is straightforward to verify that the quantities defined in (39) and (40) are invariant under homotheties of Ω. Below we characterize those pairs $\{(p, q) : p \ge 1\}$ for which the sharp constants defined in (41) and (42) are finite.

Theorem 4 (i) *For $1 \le p < \infty$, $\mathfrak{F}_{p,q} < \infty$ if and only if $q \le 1$.*
(ii) *For $p = \infty$, $\mathfrak{F}_{\infty,q} < \infty$ if and only if $q \le 1$.*

Proof (i) We first suppose $q > 1,\ 1 \le p < \infty$. Let Ω_n be the disjoint union of n balls with equal radii r_n, where $|\Omega_n| = \omega_m n r_n^m = 1$. Then $\lambda_1(\Omega_n) = r_n^{-2}\lambda_1(B_1)$. By scaling we have that

$$T_p^p(\Omega_n) = r_n^{2p}|B_1|^{-1}T_p^p(B_1). \tag{43}$$

Hence by (43),

$$\mathfrak{F}_{p,q}^p \ge F_{p,q}^p(\Omega_n) = |B_1|^{-1} r_n^{2p-2pq} T_p^p(B_1)\lambda_1^{pq}(B_1). \tag{44}$$

Since $q > 1$ and $r_n \downarrow 0$ as $n \to \infty$, we have that the right-hand side of (44) tends to infinity as $n \to \infty$.

Next suppose $q \le 1,\ 1 \le p < \infty$. By (39), Faber–Krahn, and Theorem 1

$$\begin{aligned} F_{p,q}(\Omega) &= \frac{T_p(\Omega)\lambda_1(\Omega)}{|\Omega|^{\frac{1}{p}}}\lambda_1^{q-1}(\Omega)|\Omega|^{\frac{2}{m}(q-1)} \\ &\le \mathfrak{F}_p \lambda_1^{q-1}(B_1)|B_1|^{\frac{2}{m}(q-1)} \\ &\le \mathfrak{F}_\infty \lambda_1^{q-1}(B_1)|B_1|^{\frac{2}{m}(q-1)}. \end{aligned}$$

This proves part (i).
(ii) We first suppose $q > 1$, and let Ω_n be the set as in the proof of part (i) above. Then $\|v_{\Omega_n}\|_{L^\infty(\Omega_n)} = \frac{r_n^2}{2m}$. Hence

$$\mathfrak{F}_{\infty,q} \ge \frac{r_n^{2-2q}\lambda_1^q(B_1)}{2m},$$

which tends to infinity as r_n tends to 0.

Next suppose $q \le 1$. By (40), (4) and Faber–Krahn,

$$\begin{aligned} F_{\infty,q}(\Omega) &= \frac{\|v_\Omega\|_{L^\infty(\Omega)}\lambda_1^q(\Omega)}{|\Omega|^{\frac{2}{m}(1-q)}} \\ &\le \mathfrak{F}_\infty \lambda_1^{q-1}(\Omega)|\Omega|^{\frac{2}{m}(q-1)} \\ &\le \mathfrak{F}_\infty \lambda_1^{q-1}(B_1)|B_1|^{\frac{2}{m}(q-1)}. \end{aligned}$$

This proves part (ii). □

In general it looks very difficult to compute $\mathfrak{F}_{p,q}$ or even $\mathfrak{F}_p = \mathfrak{F}_{p,1},\ p > 2$, with the exception of $\mathfrak{F}_{p,0}$. G. Talenti in [15] obtained a pointwise estimate between the rearrangement of the torsion function of a generic set with finite measure and the torsion function of the ball with the same measure. In particular this estimate implies that the L^p norm of the torsion function is maximised by the L^p norm of the torsion function for the ball with the same measure. Hence, by (39) and (40) we have

$$\mathfrak{F}_{p,0} = \frac{T_p(B_1)}{|B_1|^{\frac{1}{p}+\frac{2}{m}}}.$$

However, in the one-dimensional case we have the following result.

Theorem 5 *If* $m = 1,\ q \le 1,\ 1 \le p < \infty$, *then*

$$\mathfrak{F}_{p,q} = \frac{\pi^{(4pq+1)/(2p)}}{2^{(1+3p)/p}} \left(\frac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})} \right)^{1/p}, \tag{45}$$

and

$$\mathfrak{F}_{\infty,q} = \frac{\pi^{2q}}{8}. \tag{46}$$

Proof of Theorem 5 Since $\Omega \subset \mathbb{R}^1$ is open it is a countable union of open intervals. Since $|\Omega| < \infty$, we let $2a_1 \ge 2a_2 \ge \cdots$ be the lengths of these intervals. Without loss of generality we may assume that $|\Omega| = 2\sum_{j=1}^{\infty} a_j = 1$. By the first equality in (25) we have by scaling for a single interval B_a of length $2a$ that

$$\begin{aligned} T_p(B_a) &= \frac{a^{(2p+1)/p}}{2} \left(2 \int_{[0,1]} dr\, (1-r^2)^p \right)^{1/p} \\ &= \frac{a^{(2p+1)/p} \pi^{1/(2p)}}{2} \left(\frac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})} \right)^{1/p} \\ &= a^{(2p+1)/p} c_p, \end{aligned} \tag{47}$$

where c_p can be read-off from (47). Since T_p^p is additive on disjoint open sets we have that

$$T_p^p(\Omega) = c_p^p \sum_{j=1}^{\infty} a_j^{2p+1} \le c_p^p a_1^{2p} \sum_{j=1}^{\infty} a_j = 2^{-1} c_p^p a_1^{2p}.$$

Since

$$\lambda_1(\Omega) = \frac{\pi^2}{4a_1^2},$$

$q \le 1$, and $2a_1 \le 1$, we have that $(2a_1)^{2-2q} \le 1$. Hence

$$F_{p,q}(\Omega) \le 2^{-1/p} c_p \left(\frac{\pi^2}{4} \right)^q a_1^{2-2q} \le 2^{-(1+2p)/p} \pi^{2q} c_p.$$

By taking the supremum over all $\Omega \subset \mathbb{R}^1$ with measure 1 we obtain that

$$\mathfrak{F}_{p,q} \le 2^{-(1+2p)/p} c_p \pi^{2q}. \tag{48}$$

To obtain a lower bound for $\mathfrak{F}_{p,q}$ we make the particular choice of $\Omega = B_1$. This gives that

$$\mathfrak{F}_{p,q} \ge F_{p,q}(B_1) = 2^{-(1+2p)/p} \pi^{2q} c_p. \tag{49}$$

By (48) and (49) we conclude that

$$\mathfrak{F}_{p,q} = F_{p,q}(B_1) = 2^{-(1+2p)/p}\pi^{2q}c_p. \tag{50}$$

and (45) follows from (50) and the definition of c_p in (47).

To prove (46) we just observe that the maximum of the torsion function and the first Dirichlet eigenvalue are determined by the largest interval in Ω, i.e. a_1. Since $q \le 1$ we maximise the resulting expression by taking $a_1 = \frac{1}{2}$. $\square$

Note that as B_1 is convex we also have that

$$\mathfrak{F}_p^{\text{convex}} = \mathfrak{F}_{p,1} = \mathfrak{F}_p, \tag{51}$$

and recover the known values $\mathfrak{F}_1 = \frac{\pi^2}{12}, \mathfrak{F}_\infty = \frac{\pi^2}{8}$ [4,5]. Note that $\mathfrak{F}_1 < \mathfrak{F}_2 = \frac{\pi^2}{\sqrt{120}} < 1$, which is in contrast with the higher dimensional situation $m \ge 2$, where $\mathfrak{F}_p = 1,\ 1 \le p \le 2$. $\square$

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